How airplanes fly and ships sail

The discussion here is a supplement of calculations and pictures to Eremenko's note `https://www.math.purdue.edu/~eremenko/dvi/airplanes.pdf`, which has more information and references. See also sections 2.1.1, 3.4.1, and 4.2 of Fisher's *Complex Variables*.

Let $v(z)$ be a vector field (representing the wind) which is incompressible (i.e. divergence free, $\partial_x \operatorname{Re} v + \partial_y \operatorname{Im} v = 0$) and irrotational (i.e. curl free, $\partial_x \operatorname{Im} v - \partial_y \operatorname{Re} v = 0$).

This is the nicest kind of fluid flow, with no vortices, turbulence, viscosity, etc. Air can behave like this under favorable conditions. We are interested in flow around an impermeable object. We represent the object by D, a domain in the complex plane. Impermeability means that v is defined on the exterior of D and is tangent to the boundary of D. Thus there is no drag; neglecting drag makes sense for a sufficiently aerodynamic object, such as an airplane wing or a taut sail nearly parallel to the wind.

A basic example, which is a building block of the more complicated examples below, is the circular vector field $v(z) = -ic/\bar{z}$ with c real, where D is the disk $\{z\colon |z| < R\}$ for some $R > 0$. See Figure 1.

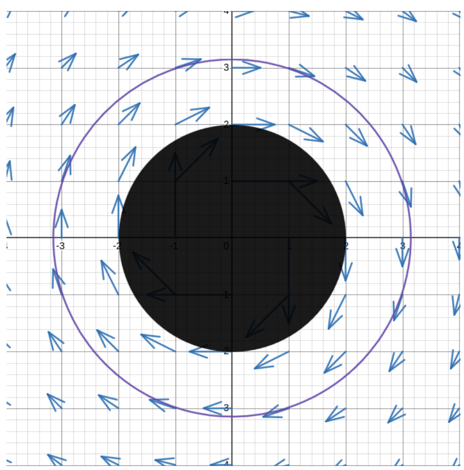

FIGURE 1. A circular flow around a circular object: $D = \{z\colon |z| < 2\}$, $v(z) = -ic/\bar{z}$, $f(z) = ic/z$, $F(z) = ic \log z$, with $c = 15$. The purple circle is the level set $\operatorname{Im} F(z) = 17.2$. You can adjust the parameters here: `https://www.desmos.com/calculator/sd2pssl2gx`.

When $v(z)$ is incompressible and irrotational, the function f defined by $f(z) = \overline{v(z)}$ is analytic because f obeys the Cauchy–Riemann equations. The function f is called the *complex velocity*.

Let F be a complex antiderivative of f, so $F'(z) = f(z)$. The function F is called the *complex potential*. It is significant because for every z, $v(z)$ is tangent to the level set of $\operatorname{Im} F$ at z. To

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prove this, notice that if $\gamma'(t) = v(\gamma(t))$ (so γ is a parametrized curve representing the trajectory of a particle in the wind) then

$$\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))v(\gamma(t)) = |f(\gamma(t))|^2.$$

Taking imaginary part of both sides gives $\frac{d}{dt}\operatorname{Im} F(\gamma(t)) = 0$, so each γ lives on a level set of $\operatorname{Im} F$ and that means that for every z, $v(z)$ is tangent to the level set of $\operatorname{Im} F$ at z.

Now we are ready to treat the fundamental example we began above more fully.

The case of a circle. We start with the case where D is a disk. Let $v_\infty > 0$ be the background velocity of the wind (i.e. the velocity far away from D), and c be a real number which will measure circulation around D. We will check that if

$$f(z) = v_\infty + \frac{ic}{z} - \frac{v_\infty R^2}{z^2}, \tag{1}$$

then $v(z) = \overline{f(z)}$ is tangent to the circle $|z| = R$. To check this, note that we are checking that $v(z)$ is perpendicular to z when $|z| = R$, i.e. as on page 8 of Fisher that $\operatorname{Re} z\overline{v(z)} = \operatorname{Re} zf(z) = 0$. So we calculate

$$\operatorname{Re} zf(z) = \operatorname{Re} v_\infty z + ic - \frac{v_\infty R^2}{z} = v_\infty(\operatorname{Re} z) - v_\infty R^2(\operatorname{Re} z)/|z|^2,$$

which is 0 when $|z| = R$. Another way to check the tangency requirement is to calculate

$$F(z) = v_\infty z + ic\log z + v_\infty R^2/z, \qquad \operatorname{Im} F(z) = v_\infty \operatorname{Im} z + ic\log|z| - v_\infty \operatorname{Im} zR^2/|z|^2,$$

and note that if $|z| = R$ then $\operatorname{Im} F(z) = c\log|R|$ which is independent of z. Thus the circle $|z| = R$ is contained in the level set $\operatorname{Im} F(z) = c\log|R|$ and since v is tangent to the level set it is also tangent to the circle. See Figure 2.

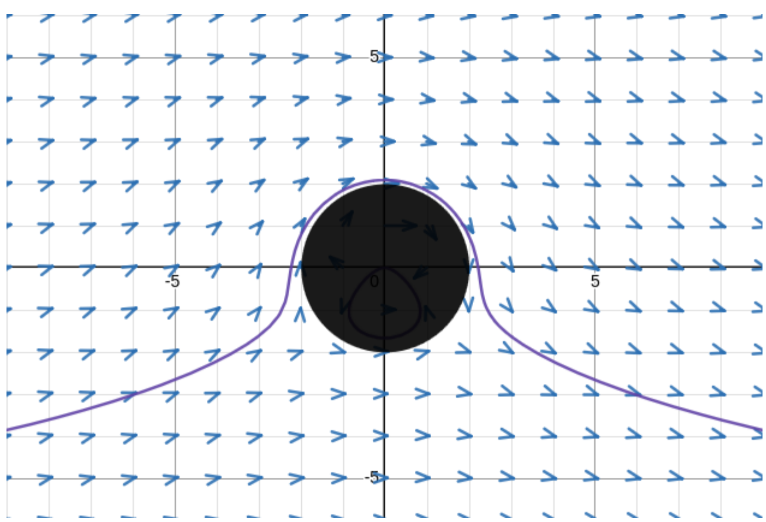

FIGURE 2. A plot when $v_\infty = 1$, $c = 2.5$, and $R = 2$. The purple curve is $\operatorname{Im} F(z) = 2$ and depicts a flowline going very close to the top of the wing. You can adjust the parameters here: `https://www.desmos.com/calculator/bxv6wo7upp`.

It turns out that the above are *all* the possibilities for f. More precisely, given constants $v_\infty > 0$ and $R > 0$, if 1) f is analytic on $\{z\colon |z| > R\}$, 2) f obeys $\lim_{|z|\to\infty} f(z) = v_\infty$, and 3) $v(z) = \overline{f(z)}$ is tangent to $|z| = R$ (i.e. we have $\operatorname{Re} z f(z) = 0$ when $|z| = R$) then there is a real c such that f is given by (1). This will be explained below under 'Uniqueness'.

Other cross sections. We can treat many other cross sections D by using a mapping, or change of variables, or change of coordinates, to reduce to the case of a circle. The most general result of this kind is the Riemann mapping theorem.[1] See the rest of Chapter 3 of Fisher for more discussion and various examples and general methods. We will just look at a few examples.

The case of a segment. Let D be given by the segment from $-Le^{-i\alpha}$ to $Le^{i\alpha}$ for some real L and α. We start with the case $L = 2$, $\alpha = 0$, for which we use the mapping $z \mapsto w(z)$ defined by

$$z = w(z) + \frac{1}{w(z)}.$$

This is called the *Joukowski* mapping. To see that the exterior regions $\{z\colon \operatorname{Im} z \neq 0 \text{ or } |\operatorname{Re} z| > 2\}$ and $\{w\colon |w| > 1\}$ are in one-to-one correspondence, note that

$$\operatorname{Re} z = \operatorname{Re} w(1 + |w|^{-2}), \qquad \operatorname{Im} z = \operatorname{Im} w(1 - |w|^{-2}).$$

and so if $R > 1$ then the circle $|w| = R$ is mapped to the ellipse passing through the points $\pm(R + R^{-1})$ and $\pm i(R - R^{-1})$. See Figure 3.

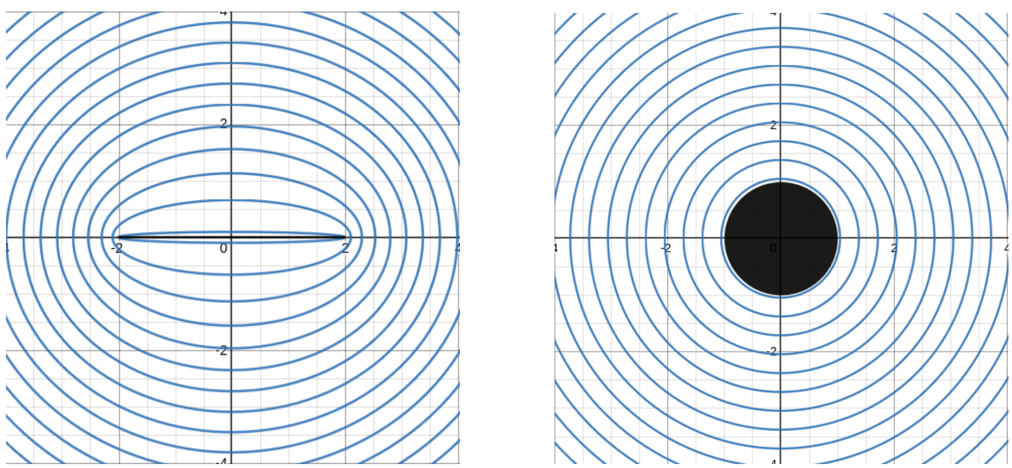

Figure 3. The correspondence $z = w + \frac{1}{w}$. The circles $|w| = R$ for various values of $R > 1$ on the right, with the respective ellipses on the left. Thus the exterior regions $\{z\colon \operatorname{Im} z \neq 0 \text{ or } |\operatorname{Re} z| > 2\}$ and $\{w\colon |w| > 1\}$ are in one-to-one correspondence. See `https://www.desmos.com/calculator/kqeypion7o` and `https://www.desmos.com/calculator/1hdmplt0xa`.

We can use as a complex potential the function

$$F(z) = v_\infty w(z) + ic \log w(z) + v_\infty / w(z) = v_\infty z + ic \log z + \cdots, \tag{2}$$

[1] Basically, this says that it can be done as long as D is a connected, simply connected, bounded, open set. But there are some issues to resolve in cases where the boundary of D is not smooth.

where the first equals sign is the definition of F, and for the second we used $w(z) - z \to 0$ as $z \to \infty$; the $\cdots$ in (2) is a bounded analytic function, defined in the complement of D, with a series $\sum_{n=0}^{\infty} a_n z^{-n}$ that we do not need to compute.[2]

For the more general segment from $-Le^{-i\alpha}$ to $Le^{i\alpha}$, we multiply z by $2e^{i\alpha}/L$ to map back to the segment from -2 to 2, and give w the same factor so that we maintain $w(z) - z \to 0$ as $z \to \infty$. That gives

$$\frac{2e^{i\alpha}z}{L} = \frac{2e^{i\alpha}w(z)}{L} + \frac{L}{2e^{i\alpha}w(z)}, \qquad \text{or} \qquad z = w(z) + \frac{L^2}{4e^{2i\alpha}w(z)}. \tag{3}$$

We once again get F of the form (2), but now with w given by (3).

Lift force and Kutta's principle. For any shape of sail/wing, if $f(z) = v_\infty + \frac{ic}{z} + \cdots$ (or equivalently $F(z) = v_\infty z + ic\log z + \cdots$) then the lift force on the wing can be derived from Bernoulli's principle[3] and it is

$$\overline{\frac{i\rho}{2}\int_{\partial D} f(z)^2 dz} = 2\pi\rho v_\infty ci,$$

where ρ is the density of the fluid. To determine c, we use *Kutta's principle*, which says that if there is a trailing sharp edge, then $f(z) = 0$ there so that the flow leaves the edge smoothly. In the case above, the trailing sharp edge is at $z = Le^{-i\alpha}$, i.e. $\frac{2e^{i\alpha}w(z)}{L} = 1$. To compute c, we substitute $w = Le^{-i\alpha}/2$ and $R = |w| = L/2$ into $v_\infty + \frac{ic}{w} + \frac{R^2}{w^2} = 0$ and solve for c to get $c = v_\infty L \sin\alpha$. Thus the magnitude of the lift is

$$2\pi\rho v_\infty^2 L \sin\alpha.$$

Let's say for instance we have $\rho = 1\,kg/m^2$ (the density of air) $v_\infty = 10\,m/s$ (a nice breeze of about 22 miles per hour) $L = 1/2\,m$ (the total segment is one meter), $\sin\alpha = 1/10$ (our angle with the wind is about $5.7°$). That gives 10π Newtons of force or about 15 pounds per square meter of sail. If the sail is 10 square meters that gives about 150 pounds.

Note that we are neglecting drag, so α must be small for this to be realistic.

The case of a Joukowski airfoil. We obtain a *Joukowski airfoil*, which is a classic airplane wing cross section, by using again $z = w + \frac{1}{w}$ but replacing the circle $|w| = 1$ with a circle $|w - p| = |1 - p|$, with p close to the origin: see Figure 4.

Since $w(2) = 1$, the trailing sharp edge at $z = 2$ is mapped to the point 1 on the circle. If we parametrize the circle with $w = p + |1-p|e^{it}$, then $w = 1$ corresponds to $e^{it} = (1-p)/|1-p|$, so $t = \arg(1-p)$. This corresponds to the case of a tilted segment with $\alpha = -t$, $\sin\alpha = \operatorname{Im} p/|1-p|$, and $L = 2R = 2|1-p|$. Thus the magnitude of the lift force is

$$4\pi\rho v_\infty^2 \operatorname{Im} p.$$

[2] To compute F more explicitly, use the quadratic formula to solve $z = w(z) + 1/w(z)$ by writing $w(z)^2 - zw(z) + 1 = 0$, $w(z) = (z/2) \pm \sqrt{(z^2/4) - 1} = (z/2)(1 + \sqrt{1 - 4/z^2})$, where we are using the principal branch of the square root; to check the branch, note that if z is in $[-2, 2]$, then $z^2/4$ is in $[0, 1]$, $4/z^2$ is in $[1, \infty)$, and so $1 - 4/z^2$ is in $(-\infty, 0]$, which is where the cut of the principal branch of the square root goes.

[3] See the first few pages of `https://www.math.purdue.edu/~eremenko/dvi/airplanes.pdf`.

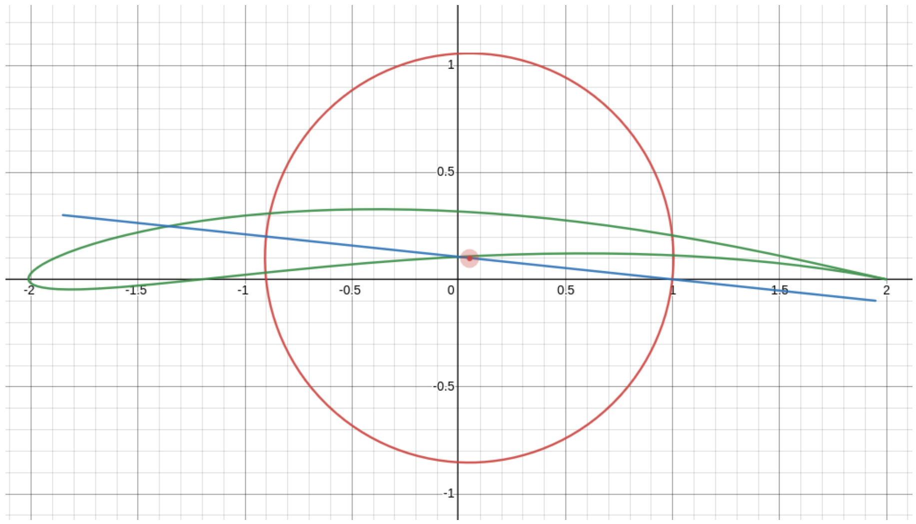

FIGURE 4. Green is the Joukowski airfoil corresponding to the red circle, which is centered at $p = (1+2i)/20$ and passes through 1. Blue is a segment which has the same lift as the green. You can adjust p to get different Joukowski airfoils here: `https://www.desmos.com/calculator/0pgrffm6zf`.

Uniqueness. Let's check that, given constants v_∞ and $R > 0$, with equation (1) we have found *all* the functions f such that

i) f is analytic on $\{z\colon |z| > R\}$ and continuous on $\{z\colon |z| \geq R\}$,
ii) $\lim_{|z|\to\infty} f(z) = v_\infty$, and
iii) $\operatorname{Re} zf(z) = 0$ when $|z| = R$.

This shows that the discussion of 'The case of a circle' above covers *all* examples with $D = \{z\colon |z| < R\}$.

Start with an arbitrary such f. Then on $\{z\colon |z| > R\}$, f has a Laurent series expansion

$$f(z) = v_\infty + \sum_{n=1}^{\infty} a_n z^{-n}.$$

We now consider the difference

$$g(z) = f(z) - \left(v_\infty + \frac{ic}{z} - \frac{v_\infty R^2}{z^2}\right),$$

with $c = \operatorname{Im} a_1$. We will use a carefully chosen transformation to convert g into a function h such that

iv) h is analytic on $\{z\colon |z| < 1/R\}$ and continuous on $\{z\colon |z| \leq 1/R\}$, and
v) $\operatorname{Re} h(z) = 0$ when $|z| = 1/R$.

Then by the maximum principle (see the section below) it will follow that $\operatorname{Re} h(z) = 0$ on $\{z\colon |z| \leq 1/R\}$. By the Cauchy–Riemann equations, $\operatorname{Im} h(z) = 0$ on $\{z\colon |z| \leq 1/R\}$ as well, from which it will follow that $g(z) = 0$ on $\{z\colon |z| \geq R\}$.

To obtain h, note that

$$g(z) = \sum_{n=1}^{\infty} b_n z^{-n},$$

with b_1 real. Moreover, by direct calculation and using $\operatorname{Re} zf(z) = 0$, we have $\operatorname{Re} zg(z) = 0$ when $|z| = R$. Now let

$$h(z) = g(1/z)/z = \sum_{n=1}^{\infty} b_n z^{n-1}.$$

This h has properties iv) and v), so $h(z) = 0$ on $\{z\colon |z| \le 1/R\}$ and $g(z) = 0$ on $\{z\colon |z| \ge R\}$.

Maximum principle. The maximum principle states that if h is analytic on $\{z\colon |z| < 1/R\}$ and continuous on $\{z\colon |z| \le 1/R\}$, then $\operatorname{Re} h$ attains its maximum and minimum values on $\{z\colon |z| = 1/R\}$.

This follows from the fact that, by the Cauchy integral formula,

$$h(p) = \frac{1}{2\pi i} \int_{|z-p|=r} \frac{h(z)dz}{z-p} = \frac{1}{2\pi} \int_0^{2\pi} h(p + re^{i\theta}) d\theta, \tag{4}$$

when $|p| < 1/R$ and $r < 1/R - |p|$; equation (4) is called the *mean value property* because the value of h at p is the average of the values of h on any circle centered at p, as long as the circle fits within the zone of analyticity $|z| < 1/R$.

Let's show that the mean value property (4) implies the maximum principle. To do this, we show that if $\operatorname{Re} h$ attains a maximum at some p in $\{z\colon |z| < 1/R\}$ then $\operatorname{Re} h$ is constant, and hence it attains the same maximum on $\{z\colon |z| = 1/R\}$ as well. Note that $\operatorname{Re} h(p + re^{i\theta}) = \operatorname{Re} h(p)$ for all $r < 1/R - |p|$ and all θ because we have $\operatorname{Re} h(p + re^{i\theta}) \le \operatorname{Re} h(p)$ by the fact that $\operatorname{Re} h(p)$ is maximal and if we ever had $\operatorname{Re} h(p + re^{i\theta}) < \operatorname{Re} h(p)$ we would have $\frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} h(p + re^{i\theta}) d\theta < \operatorname{Re} h(p)$, violating (4). This shows that $\operatorname{Re} h$ is constant on $\{z\colon |z - p| < 1/R - |p|$. If we had $p = 0$ then this reduces to $\operatorname{Re} h$ is constant on $\{z\colon |z| < 1/R\}$. If $p \ne 0$, then repeat the argument with p replaced by p', where p' is closer to 0 than p. Repeating the argument enough times proves that $\operatorname{Re} h$ is constant on $\{z\colon |z| < 1/R\}$. Applying the same result with $-h$ in place of h proves that $\operatorname{Re} h(p) \ge 0$ for all p.